MATH 504 HOMEWORK 6

Due Monday, November 19.

Problem 1. *Show there is a projection* $\pi : Add(\omega, \lambda) \to Add(\omega, 1)$.

Problem 2. *Let* M *be a transitive model of* ZFC *and* $\mathbb{P} \in M$ *be a poset. Suppose that* $p \in \mathbb{P}$ *is such that* $p \Vdash$ "$\dot{f} : \lambda \to \tau$ *is a function*".

(1) *Show that for every* $\alpha < \lambda$, $\{q \mid \exists \gamma \in \tau (q \Vdash \dot{f}(\alpha) = \gamma)\}$ *is dense below* p.
(2) *Let* $B = \{\gamma < \tau \mid (\exists q \leq p)(\exists \alpha < \lambda)(q \Vdash \dot{f}(\alpha) = \gamma)\}$. *Show that if* $\sup(B) < \tau$, *then* $p \Vdash$ "$\dot{f}$ *is bounded*".

We say that $\mathbb{P}$ preserves cofinalities if for every ordinal α, if in V, $\mathrm{cf}(\alpha) = \tau$, then $1_{\mathbb{P}} \Vdash \mathrm{cf}(\alpha) = \tau$.

Problem 3. *Prove (in detail) that if* $\mathbb{P}$ *preserves cofinalities, then* $\mathbb{P}$ *preserves cardinals.*

Problem 4. *Suppose* $\mathbb{P}$ *and* $\mathbb{Q}$ *are two posets and* $i : \mathbb{P} \to \mathbb{Q}$ *is such that:*

(1) $i(1_{\mathbb{P}}) = 1_{\mathbb{Q}}$;
(2) *If* $p' \leq_{\mathbb{P}} p$, *then* $i(p') \leq_{\mathbb{Q}} i(p)$;
(3) *For all* $p_1, p_2 \in \mathbb{P}$, $p_1 \perp p_2$ *iff* $i(p_1) \perp i(p_2)$;
(4) *If* A *is a maximal antichain of* $\mathbb{P}$, *then* $i``A := \{i(p) \mid p \in A\}$ *is a maximal antichain in* $\mathbb{Q}$.

Suppose also that H *is* $\mathbb{Q}$*-generic. Show that* $G := \{p \in \mathbb{P} \mid i(p) \in H\}$ *is* $\mathbb{P}$*-generic and that* $V[G] \subset V[H]$, *where* V *is the ground model.*

Remark: an embedding as above is called a *complete embedding*.

Problem 5. *Suppose that for all* n, $2^{\aleph_n} = \aleph_{\omega+1}$. *Show that* $2^{\aleph_\omega} = \aleph_{\omega+1}$.

Hint: For each $A \subset \aleph_\omega$, *define* $A_n := A \cap \aleph_n$. *Consider the map* $A \mapsto \langle A_n \mid n < \omega \rangle$.

Problem 6. *Suppose that* $\mathbb{P}$ *is a poset,* $A \subset \mathbb{P}$ *is a maximal antichain,* $\phi(x)$ *is a formula, and* $\langle \tau_p \mid p \in A \rangle$ *are* $\mathbb{P}$ *names, such that for all* $p \in A$, $p \Vdash \phi(\tau_p)$. *Show that there is a* $\mathbb{P}$ *name* τ, *such that* $1_{\mathbb{P}} \Vdash \phi(\tau)$.